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Projection operator approach to constrained systems

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Abstract. Recently, having reconsidered the reproducing kernel for gauge-invariant states which involves the projection operator onto the reduced Hilbert space of physical states, John Klauder has shown how the phase space coherent state path integral quantization of constrained systems avoids any gauge-fixing conditions, and leads to a specific measure for the integration over Lagrange multipliers. Here, it is pointed out that independently of the coherent state formulation, this approach is also devoid of any Gribov problems and always provides for an effectively admissible integration over all gauge orbits of gauge-invariant systems. This important aspect of Klauder's reappraisal of the physical reproducing kernel is explicitly confirmed by two simple examples.

1. Introduction

In a recent paper [1], John Klauder considered the quantization of constrained systems within the context of phase space coherent states [2], reaching an important conclusion with regards to the path integral measure for the Lagrange multipliers which are usually introduced in order to enforce constraints. Based on the projection operator [3–7] onto the reduced Hilbert space of physical states, Klauder's analysis does not require gauge-fixing conditions for first-class constraints, nor Dirac brackets to reduce for second-class constraints, thereby avoiding the otherwise necessary consideration of potential Gribov problems [8, 9] or loss of manifest covariance under specific symmetries of the system, as well as the introduction of δ-functionals and functional determinants into path integral representations. These latter issues are typical of the conventional approaches [7, 10, 11] to the quantization of constrained systems, namely Faddeev's reduced phase space approach [12], Dirac's quantization [13] or the powerful BFV-BRST methods [14]. Nevertheless, by construction, the analysis as advocated by Klauder must lead to gauge-invariant observables to which each of the gauge equivalence classes of the possible configurations of the system can contribute once and only once. This is to be constrasted with the situation in the conventional approaches for which such a result is achieved only for 'admissible' gauge-fixing conditions, namely those which are free of Gribov problems [8, 9, 11]. It is only for the gauge equivalence class of admissible gauge-fixing conditions that the correct gauge-invariant result is obtained in the conventional approaches [11, 15, 16].

It is obviously important to provide explicit examples confirming the analysis as advocated by Klauder for the reproducing kernel or propagator of physical gauge-invariant states. This may be done by comparing the expressions to which that analysis leads to

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well established results in the case of some constrained systems. Klauder's approach [1] emphasizes the path integral representation of quantum amplitudes using phase space coherent states. In the present paper, the conclusions reached in [1] are abstracted from the specific context of phase space coherent states, and are considered from the operator point of view. Specifically, the fact that the point of view as advocated by Klauder, based on the physical projection operator, avoids the necessity of gauge fixing but nevertheless leads to gauge-invariant results associated with what would be an admissible choice of gauge fixing in the conventional approaches (whether such a choice is possible or not), is checked explicitly by way of two simple examples, for which fully satisfactory results are obtained. Such a conclusion, which must hold in general, is only implicit in [1].

The outline of the paper is as follows. In the section 2, the physical projector onto the reduced Hilbert space, which plays such a prominent role in Klauder's analysis, is briefly described in the operator context. Sections 3 and 4 then apply the general discussion to two examples in Minkowski spacetime, namely the free relativistic scalar particle and pure Yang–Mills theory in $0+1$ dimensions. Finally, some additional comments are presented in section 5.

2. The physical projector and physical propagator

Klauder's analysis [1] within the coherent state approach to quantization involves the projection operator [3–7] $\mathcal{E}$ onto the subspace of states annihilated by the constraints, namely the reduced Hilbert space of physical states. This operator may be introduced in the following way. Although the discussion can be extended to more general situations [1], for the sake of simplicity let us consider a constrained system with Grassmann even degrees of freedom and first-class constraints only whose algebra is closed [7, 11]. Phase space degrees of freedom (q^n, p_n) take values over the entire real line and possess the canonical Poisson bracket structure. The closed algebra of the first-class constraints $\phi_\alpha(q, p)$, together with the first-class Hamiltonian $H_0(q, p)$, is given by

$$\left\{\phi_\alpha(q,p), \phi_\beta(q,p)\right\} = C_{\alpha\beta}{}^{\gamma}\, \phi_\gamma(q,p) \qquad \{H_0(q,p), \phi_\alpha(q,p)\} = C_\alpha{}^{\beta}\, \phi_\beta(q,p)\,. \tag{1}$$

Here, $C_{\alpha\beta}{}^{\gamma}$ and $C_\alpha{}^{\beta}$ are specific constant structure coefficients which determine the closed algebra of connected local Hamiltonian gauge transformations of the system.

Consequently, time evolution of the system follows from the first-order action

$$S = \int \mathrm{d}t\ \left[\dot{q}^n p_n - H_T(q,p)\right] \tag{2}$$

the total Hamiltonian being given by

$$H_T(q,p;\lambda) = H_0(q,p) + \lambda^\alpha\, \phi_\alpha(q,p) \tag{3}$$

where the quantities $\lambda^\alpha(t)$ are arbitrary time dependent Lagrange multipliers for the first-class constraints. These Lagrange multipliers parametrize the local Hamiltonian gauge freedom of the system associated with the constraints. In particular, local Hamiltonian gauge transformations are given by

$$\begin{aligned} \delta_\epsilon\, q^n &= \left\{q^n\,,\ \phi_\epsilon(q,p)\right\} \\ \delta_\epsilon\, p_n &= \{p_n\,,\ \phi_\epsilon(q,p)\} \\ \delta_\epsilon\, \lambda^\alpha &= \dot{\epsilon}^\alpha + \lambda^\gamma \epsilon^\beta\, C_{\beta\gamma}{}^{\alpha} - \epsilon^\beta\, C_\beta{}^{\alpha} \end{aligned} \tag{4}$$

where the gauge generator is defined in terms of infinitesimal functions $\epsilon^\alpha(t)$ by the combination $\phi_\epsilon(q,p) = \epsilon^\alpha\, \phi_\alpha(q,p)$. These transformations provide the basis for an analysis of the space of gauge orbits of the system in its Hamiltonian formulation, and for a discussion of the possibility of admissible gauge-fixing conditions, or otherwise, of Gribov problems either of the first or second type, or both [11]. Such issues must be addressed on a case-by-case basis.

Let us now consider the quantized system. Namely, let us assume that a choice of quantum operator ordering and of inner product on the space of states is possible such that the quantum algebra of constraints between themselves and with the Hamiltonian retains the same form (1) as at the classical level, and such that quantum observables obey the appropriate self-adjoint properties. With the quantum system defined from its classical counterpart in this manner, physical or gauge-invariant states (at least invariant under those gauge transformations continuously connected to the identity transformation) are defined by the condition

$$\hat{\phi}_\alpha\,|\text{physical}\rangle = 0\,. \tag{5}$$

Time evolution of the system is induced by the total quantum Hamiltonian $\hat{H}_T$ via the time-ordered propagator†

$$S(t_2, t_1) = T\exp\left(-\mathrm{i}\int_{t_1}^{t_2}\mathrm{d}t\;\hat{H}_T\right) \tag{6}$$

which thus involves the arbitrary time dependent functions $\lambda^\alpha(t)$. Obviously, the action of the total *Hamiltonian* $\hat{H}_T$ on any physical state leads to another physical state which is independent of the choice of Lagrange multipliers $\lambda^\alpha(t)$, this being not necessarily a property shared by the propagator itself. Consequently, the evolution operator $S(t_2, t_1)$ also propagates gauge *variant* or unphysical states in a gauge dependent manner.

In order to construct a propagator for physical states only, let us consider [3–7] the projection operator onto the subspace of states annihilated by the first-class quantum constraints. Denoting this operator by $\mathcal{E}$, with the properties

$$\mathcal{E}^2 = \mathcal{E} \qquad \mathcal{E}^\dagger = \mathcal{E} \tag{7}$$

the physical projector is given by [1]‡

$$\mathcal{E} = \int \mathrm{d}U(\theta^\alpha)\;\exp\left(-\mathrm{i}\theta^\alpha\hat{\phi}_\alpha\right) \tag{8}$$

where $\mathrm{d}U(\theta^\alpha)$ is a suitable integration measure over the space of transformations generated by the first-class constraints, such that $\mathcal{E}$ does possess the properties in (7). In particular, note how the condition $\mathcal{E}^2 = \mathcal{E}$ determines the normalization of the integration measure $\mathrm{d}U(\theta^\alpha)$. For example, if these constraints generate a compact Lie group, $\mathrm{d}U$ is the associated normalized Haar measure over that group§.

Given the physical projector $\mathcal{E}$, the *physical propagator* for gauge-invariant states is then constructed to be [1, 5–7]‖

$$S_{\text{phys}}(t_2, t_1) = \exp\left(-\mathrm{i}\hat{H}_0(t_2 - t_1)\right)\mathcal{E}\,. \tag{9}$$

† Units such that $\hbar = 1$ are assumed throughout.

‡ When the spectrum of the constraints $\hat{\phi}_\alpha$ is continuous, a proper definition of the reduced physical Hilbert space requires some form of the δ-limiting procedure discussed for example in [1].

§ To be precise, first-class constraints generate only the connected component of the Lie group, whereas the full gauge group of the system may be different from its universal covering group. Such a situation may properly be implemented by appropriatedly modifying the integration domain over the group parameters θ^α in the definition of the projector $\mathcal{E}$.

‖ Note that this construction is reminiscent of Feynman's tree theorem [17].

Since the first-class constraints form a closed algebra among themselves and with the canonical Hamiltonian $\hat{H}_0$, note that one may also write

$$S_{\rm phys}(t_2,t_1)=\mathcal{E}\ \exp\big(-{\rm i}\hat{H}_0(t_2-t_1)\big)\,\mathcal{E}=\mathcal{E}\ \exp\big(-{\rm i}\mathcal{E}\hat{H}_0\mathcal{E}(t_2-t_1)\big)\,\mathcal{E}\,. \qquad (10)$$

In particular, the latter of these two expressions is the one that is relevant more generally for systems which include second-class constraints as well [1]. In this form, it should be clear that the physical propagator does indeed propagate as intermediate states physical states only, and as external states their gauge-invariant components only. Moreover, this propagator obeys [1] the convolution property required of an evolution operator

$$S_{\rm phys}(t_3,t_2)\,S_{\rm phys}(t_2,t_1)=S_{\rm phys}(t_3,t_1)\,. \qquad (11)$$

Once the choice of physical evolution operator is specified, it is of course possible to compute its matrix elements for different choices of quantum states. The latter may include, for example, configuration space eigenstates, momentum space eigenstates, or phase space coherent states. Whatever the choice, it is then also possible to develop a path integral representation of such matrix elements in the usual manner, by inserting resolutions of the identity operator $\mathbb{1}$ in terms of the chosen set of states in a stepwise discretized version of the evolution operator. Since the quantized system is assumed to have been completely defined at the operator level, including the projection operator $\mathcal{E}$, one obviously obtains a path integral representation in which the measure of all phase space degrees of freedom and Lagrange multiplier variables is uniquely determined and well defined. In particular, the property $\mathcal{E}^2=\mathcal{E}$ in (7) of the physical projector $\mathcal{E}$ uniquely determines the integration measure over the Lagrange multipliers in a path integral representation of matrix elements of the physical evolution operator [1].

This is achieved in spite of the absence of any choice of gauge fixing, thereby avoiding any potential Gribov problems in the evaluation of quantities which are gauge-invariant observables by construction. Indeed in the conventional approaches, even though gauge fixing can be effected in a manner which necessarily ensures the gauge invariance of expressions, nevertheless it is only for admissible gauge-fixing conditions that physically consistent results are obtained for gauge-invariant observables.

In contrast, the choice of physical evolution operator in (9) avoids any such difficulties at once. No choice of gauge-fixing condition has to be effected, hence no issue of a possible Gribov problem can arise. Nevertheless, gauge-invariant results are obtained, owing to the physical projector $\mathcal{E}$, by properly integrating over the space of gauge transformations. Moreover, not only does one obtain gauge-invariant results, but in addition these results must necessarily be such as to include properly the contribution of each of the gauge inequivalent configurations of the system once and only once. There is no need to go into the development of a BRST-invariant approach in order to maintain a formulation of the system which is both at the same time manifestly gauge invariant and covariant under other specific symmetries.

In [1], emphasizing the path integral point of view within the phase space coherent state approach, Klauder illustrated through a series of examples how the projector property of the operator $\mathcal{E}$ does indeed determine the path integral measure over the Lagrange multipliers. In the present paper, and within the abstract operator approach, it is the absence of Gribov problems and the admissibility of the effective integration over the space of gauge orbits of such gauge-invariant systems which are pointed out, and illustrated explicitly by way of two simple examples. Indeed, these important facts must again result from the properties of the physical projector $\mathcal{E}$.

3. The relativistic scalar particle

Consider the free relativistic scalar particle of mass $m \geqslant 0$ propagating in a Minkowski spacetime of D dimensions. The manifestly reparametrization invariant Hamiltonian formulation of this system is well known [18]. Using the notations and spacetime metric conventions of [11], with in addition a choice of units such that $c = 1$, the canonically conjugate degrees of freedom of the system are the spacetime coordinates $x^\mu(\tau)$ and energy-momentum $P^\mu(\tau)$ $(\mu = 0, 1, \ldots, D-1)$ of the particle, the canonical Hamiltonian H_0 vanishes identically as befits a reparametrization invariant dynamics, and the first-class constraint related to the connected gauge invariance of the system under orientation preserving reparametrizations of the world-line coordinate τ is

$$\phi = \tfrac{1}{2}\left[P^2 + m^2\right] . \tag{12}$$

Consequently, the total Hamiltonian of the system is simply

$$H_T = \lambda\,\phi = \tfrac{1}{2}\lambda\left[P^2 + m^2\right] \tag{13}$$

where $\lambda(\tau)$ is the Lagrange multiplier associated with the connected Hamiltonian gauge freedom generated by ϕ.

It may be shown [11] that the space of gauge inequivalent configurations of the system is characterized by the world-line metric Teichmüller parameter γ defined by

$$\gamma = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\ \lambda(\tau) \tag{14}$$

where the interval $[\tau_1, \tau_2]$ is related to a choice of boundary conditions. In particular, the parameter γ is invariant under the orientation preserving reparametrizations of the world-line, i.e. the connected gauge transformations of the system, generated by the first-class constraint ϕ. Under orientation reversing reparametrizations however, the Teichmüller parameter changes sign. Therefore, when describing the oriented scalar particle invariant under both classes of transformations, corresponding to a particle distinct from its antiparticle, the Teichmüller parameter must be restricted to a fundamental domain of the modular group [11], say the interval $\gamma \in [0, +\infty[$.

Quantization of this system is straightforward enough. One has the fundamental operator degrees of freedom $\hat{x}^\mu$ and $\hat{P}_\mu$ $(\mu = 0, 1, \ldots, D-1)$ with the canonical commutation relations

$$\left[\hat{x}^\mu\,,\ \hat{P}_\nu\right] = \mathrm{i}\delta^\mu_\nu\,. \tag{15}$$

The first-class quantum constraint is simply

$$\hat{\phi} = \tfrac{1}{2}\left[\hat{P}^2 + m^2\right] \tag{16}$$

while the generator of time evolution is the total quantum Hamitonian

$$\hat{H}_T = \lambda\,\hat{\phi} = \tfrac{1}{2}\,\lambda\left[\hat{P}^2 + m^2\right]. \tag{17}$$

Since the first-class Hamiltonian $\hat{H}_0$ vanishes identically for this system, the physical time evolution operator of the system simply reduces to the projection operator $\mathcal{E}$ itself, which in the present case is defined by

$$S_{\text{phys}}(\tau_f, \tau_i) = \mathcal{E} = \int_{-\infty}^{+\infty} \mathrm{d}\gamma\ \exp\!\left(-\tfrac{1}{2}\mathrm{i}\gamma(\hat{P}^2 + m^2)\right) \frac{\sin(\delta\gamma)}{\pi\gamma} \qquad 0 < \delta \ll 1 \tag{18}$$

with a suitable $\delta \to 0$ limit reserved to a later stage [1]. Note how the integration parameter γ is indeed to be identified with the Teichmüller parameter of the system defined in (14), on the basis of the total Hamiltonian in (17). *A priori*, the integration measure over the parameter γ could be any function of γ, since γ is invariant under local world-line reparametrizations. However, the requirements in (7) necessary for a projection operator imply in fact that the integration measure over γ should be precisely of the form as specified in (18) for some $\delta > 0$. In other words, the requirement that $\mathcal{E}$ be a projection operator essentially onto the sector of physical—or locally gauge-invariant—states effectively determines the integration measure over Teichmüller and modular space.

Given the desired projection operator, its matrix elements can be computed in a straightforward manner. Let us first consider the configuration space matrix elements, namely

$$P(x_i^\mu \to x_f^\mu) \equiv \langle x_f^\mu | \mathcal{E} | x_i^\mu \rangle \tag{19}$$

where the states $|x^\mu\rangle$ define the complete orthonormalized basis of eigenvectors of the position operators $\hat{x}^\mu$. A similar orthonormalized basis of momentum eigenstates $|p_\mu\rangle$ exists for the momentum operators $\hat{P}_\mu$. These two bases are related through the transformation rule

$$\langle p_\mu | x^\mu \rangle = (2\pi)^{-D/2} \exp\Big(-\mathrm{i} x \cdot p\Big) \tag{20}$$

in which the invariant inner product in the exponential is obviously the one defined by the Minkowski metric on spacetime. Using this rule, as well as the spectral decomposition of the identity operator $\mathbb{1}$ in terms of the momentum eigenstates $|p_\mu\rangle$, it is straightforward to obtain for the configuration space matrix elements of the physical evolution operator

$$\begin{aligned} S_{\mathrm{F}}(x_i^\mu \to x_f^\mu) &\equiv \lim_{\delta\to 0} \frac{\pi}{2\delta}\, P(x_i^\mu \to x_f^\mu) \\ &= \lim_{\delta\to 0} \frac{\pi}{2\delta} \int_{(\infty)} \frac{\mathrm{d}^D p^\mu}{(2\pi)^D}\, \exp\Big(\mathrm{i}(x_f - x_i)\cdot p\Big) \\ &\quad \times \int_{-\infty}^{+\infty} \mathrm{d}\gamma\ \exp\Big(-\tfrac{1}{2}\mathrm{i}\gamma(p^2 + m^2)\Big)\, \frac{\sin(\delta\gamma)}{\pi\gamma} \\ &= \frac{1}{2} \int_{(\infty)} \frac{\mathrm{d}^D p^\mu}{(2\pi)^D}\, \exp\Big(\mathrm{i}(x_f - x_i)\cdot p\Big) \int_{-\infty}^{+\infty} \mathrm{d}\gamma\ \exp\Big(-\tfrac{1}{2}\mathrm{i}\gamma(p^2 + m^2)\Big) \end{aligned} \tag{21}$$

where the limit $\delta \to 0$ is taken in the way discussed in [1]. The choice of normalization of the function $S_{\mathrm{F}}(x_i^\mu \to x_f^\mu)$ is such that when restricting the modular parameter γ to the range $[0, +\infty]$ (corresponding to the description of the oriented particle), the function $S_{\mathrm{F}}(x_i^\mu \to x_f^\mu)$ coincides with the Feynman propagator for the scalar particle.

Up to a constant factor, note that it is only with the integration measure over the parameter γ which appears in (21) that the Feynman propagator is obtained in that manner. Any other non-constant integration measure over γ, even though gauge invariant for local and possibly global gauge transformations (i.e for orientation preserving and reversing world-line reparametrizations, respectively), would not lead to the Feynman propagator, and would thus introduce a Gribov problem of some type [11]. In the present instance, as was pointed out above, it is precisely the fact that $\mathcal{E}$ is a projection operator with the properties in (7) which ensures the admissible integration measure over modular space, devoid of any Gribov problem. In addition, the appropriate physical propagation of gauge-invariant (i.e. reparametrization invariant) states is indeed recovered, in spite of the fact that no gauge fixing of the system is effected. Compared with the detailed calculation of the physical propagator $S_{\rm F}(x_i^\mu \to x_f^\mu)$ using Hamiltonian BRST techniques [11, 19], it is clear that the projection operator approach is far more efficient and leads immediately to the correct result [11].

In view of the basis for the analysis of [1], let us also compute the phase space coherent state† matrix elements of the physical evolution operator $\mathcal{E}$. These coherent states are defined by

$$|P_\mu, x^\mu\rangle = \exp\Big(\mathrm{i}\alpha(P_\mu, x^\mu)\Big)\ \exp\big(-\mathrm{i}x^\mu\hat{P}_\mu\big)\ \exp\Big(\mathrm{i}P_\mu\hat{x}^\mu\Big)\ |\eta\rangle \tag{22}$$

with an arbitrary phase factor $\alpha(P_\mu, x^\mu)$ and normalized fiducial state $|\eta\rangle$. It is then a simple exercise to compute the coherent state matrix elements of the projector $\mathcal{E}$:

$$\begin{aligned}\langle P_2, x_2|\mathcal{E}|P_1, x_1\rangle = \exp\Big(-\mathrm{i}\alpha(P_2, x_2)\Big)\ \exp\Big(\mathrm{i}\alpha(P_1, x_1)\Big)\\ \times\int_{(\infty)} \mathrm{d}^D p^\mu\ \exp\Big(\mathrm{i}(x_2 - x_1)\cdot p\Big)\,\eta^*(p - P_2)\,\eta(p - P_1)\\ \times\int_{-\infty}^{+\infty}\mathrm{d}\gamma\ \exp\big(-\tfrac{1}{2}\gamma(p^2 + m^2)\big)\,\frac{\sin(\delta\gamma)}{\pi\gamma}\end{aligned} \tag{23}$$

where $\eta(P_\mu)$ is the momentum space wavefunction of the fiducial state $|\eta\rangle$, namely the quantity $\eta(P_\mu) = \langle P_\mu|\eta\rangle$.

Given this expression and the resolution of the identity operator $\mathbb{1}$ in terms of the overcomplete basis of phase space coherent states, it is straightforward to verify that the configuration space matrix elements of the projection operator $\mathcal{E}$ are again given by (21), independently of the choice of fiducial state $|\eta\rangle$ used in the definition of coherent states. This check uses the relation

$$P(x_i^\mu \to x_f^\mu) = \int_{(\infty)} \frac{\mathrm{d}^D P_2 \mathrm{d}^D x_2}{(2\pi)^D}\frac{\mathrm{d}^D P_1 \mathrm{d}^D x_1}{(2\pi)^D}\ \langle x_f^\mu|P_2, x_2\rangle\langle P_2, x_2|\mathcal{E}|P_1, x_1\rangle\langle P_1, x_1|x_i^\mu\rangle \tag{24}$$

as well as the overlap functions $\langle y^\mu|P, x\rangle$ which are easily obtained from (22) and (20).

4. Pure Yang–Mills theory in $0 + 1$ dimensions

Let us consider a pure Yang–Mills theory in a Minkowski spacetime of $1 + 1$ dimensions, based on an arbitrary simple compact Lie group G of dimension D_G and of rank ℓ. The Lie algebra generators T^a ($a = 1, 2, \ldots, D_G$) obey the commutation relations

$$[T^a, T^b] = \mathrm{i}f^{abc}T^c \tag{25}$$

with real, fully antisymmetric structure coefficients f^{abc}. In particular, the adjoint representation of dimension D_G possesses the matrix representation $\big(T^a_{\rm Adj}\big)^{bc} = -\mathrm{i}f^{abc}$. The gauge vector potential components are denoted A^a_μ ($\mu = 0, 1$), while the gauge coupling constant is denoted by g, so that the gauge field strength is $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$.

† In [1], the initial example of a first-class constraint considers the motion of a particle on a hypersphere with vanishing Hamiltonian. In [1] Klauder discusses how the associated coherent states are related to the euclidian group which appears in that context. The relativistic particle is similar in that the constraint enforces the *momentum* to lie on a hypersphere of Minkowski signature. One may thus raise the question of the characterization of the group of transformations in momentum space *and in spacetime* related to the phase space coherent state matrix elements of the evolution operator for the relativistic particle, along the lines of the first example in [1, section 6].

Given these data and following the discussion of [5], let us now consider the dimensional reduction of this pure Yang–Mills theory to $0+1$ dimensions, by retaining only the ∂_1 zero modes of the fields in space, namely by assuming that the fields $A^a_\mu(t=x^0, x^1)$ are now independent of the space coordinate x^1. In order to avoid any confusion, let us then distinguish the time and space components of the gauge fields as follows:

$$\phi^a(t) = A^a_0(t) \qquad A^a(t) = A^a_1(t) \tag{26}$$

so that the only non-vanishing component of the field strength is now given by

$$F^a_{01} = \dot{A}^a + gf^{abc}\phi^b A^c\,. \tag{27}$$

Consequently, the dimensionally reduced system is described by the Lagrangian

$$L = \tfrac{1}{2}\left[\dot{A}^a + gf^{abc}\phi^b A^c\right]^2 - \tfrac{1}{2}m^2(A^a)^2 \tag{28}$$

where a gauge-invariant mass term for the A^a degrees of freedom has been added. Indeed, the reduced system possesses the following gauge invariance:

$$A'^a T^a = U\, A^a T^a\, U^{-1} \qquad \phi'^a T^a = U\, \phi^a T^a\, U^{-1} + \frac{\mathrm{i}}{g}\, U\, \frac{\mathrm{d}}{\mathrm{d}t} U^{-1} \tag{29}$$

where $U(t) = \exp\left(-\mathrm{i}g\theta^a(t)T^a\right)$ is an arbitrary time dependent transformation in G. Quite obviously, the mass term does not spoil this gauge invariance. As will become clear later on, the mass term serves the purpose of a regularization of the quantized theory.

Owing to the absence of a dependence on the time derivative of the degrees of freedom ϕ^a in the above Lagrangian, the present is a constrained system possessing a gauge invariance under the simple compact Lie group G. As a matter of fact, the full gauge invariance of the system, including connected (i.e. local) and non-connected (i.e. global) gauge transformations is not the universal covering group G_{univ} generated by the above algebra, but rather the simple compact Lie group $G = G_{\mathrm{univ}}/C$, where C is the maximal torus or center of the group G_{univ}. Indeed, the degrees of freedom ϕ^a and A^a transform under the adjoint representation of G or G_{univ}.

The Hamiltonian formulation

The usual analysis of constraints starting from the Lagrangian (28) can be applied straightforwardly; the details are not presented here. Let us only point out that the analysis follows the same lines [11] as for Yang–Mills theory in a Minkowski spacetime of dimension $(D-1)+1$, and that some of the degrees of freedom, namely the sector of the coordinates ϕ^a and their conjugate momenta, may be decoupled by considering the so-called [11] *fundamental Hamiltonian description* of the system.

In the present instance, this fundamental description is based on the phase space degrees of freedom, that is the coordinates $A^a(t)$ and their conjugate momenta $\pi^a(t)$, obeying the algebra of Poisson brackets

$$\{A^a(t), \pi^b(t)\} = \delta^{ab}\,. \tag{30}$$

The system possesses first-class constraints only, namely the gauge charges generating the local gauge transformations, which in fact also enforce Gauss' law in the present case:

$$Q^a = gf^{abc}A^b\pi^c \tag{31}$$

whose closed algebra is simply that of the Lie algebra of the gauge group G

$$\{Q^a(t), Q^b(t)\} = gf^{abc}Q^c(t)\,. \tag{32}$$

Finally, the first-class Hamiltonian is simply

$$H_0 = \tfrac{1}{2}\left(\pi^a\right)^2 + \tfrac{1}{2}m^2\left(A^a\right)^2 \tag{33}$$

so that the total Hamiltonian generating the time evolution of the system is

$$H_T = \tfrac{1}{2}\left(\pi^a\right)^2 + \tfrac{1}{2}m^2\left(A^a\right)^2 - \phi^a Q^a\,. \tag{34}$$

Here, the variables ϕ^a are *Lagrange multipliers*† for the first-class constraints Q^a, which, in fact, may be identified with the original gauge degrees of freedom as $\phi^a = A_0^a$, the latter thus also parametrizing the local gauge freedom of the system.

Given the total Hamiltonian, the Hamiltonian equations of motion are readily derived

$$\dot{A}^a = \pi^a - gf^{abc}\phi^b A^c \qquad \dot{\pi}^a = -gf^{abc}\phi^b\pi^c - m^2 A^a \tag{35}$$

whose solutions thus involve the arbitrary Lagrange multipliers ϕ^a. Similarly, local Hamiltonian gauge transformations generated by the first-class constraints Q^a read

$$\begin{aligned}\delta_\epsilon A^a &= \{A^a, Q_\epsilon\} = gf^{abc}\epsilon^b A^c\\ \delta_\epsilon\pi^a &= \{\pi^a, Q_\epsilon\} = gf^{abc}\epsilon^b\pi^c\\ \delta_\epsilon\phi^a &= -\dot{\epsilon}^a + gf^{abc}\epsilon^b\phi^c\end{aligned} \tag{36}$$

with

$$Q_\epsilon = \epsilon^a Q^a \tag{37}$$

the $\epsilon^a(t)$ being arbitrary infinitesimal functions of time. It is a straightforward exercise to check that the first-order Hamiltonian Lagrangian

$$L_{\text{Hamilt}} = \dot{A}^a\pi^a - H_T \tag{38}$$

is indeed invariant under these transformations, since

$$\delta_\epsilon(\dot{A}^a\pi^a) = \dot{\epsilon}^a Q^a \qquad \delta_\epsilon H_T = \dot{\epsilon}^a Q^a\,. \tag{39}$$

In fact, it is possible even to determine the Hamiltonian gauge transformations to all orders, and not only in linearized form. For this purpose, let us define the finite gauge transformation in the group G:

$$U(t) = \exp\left(-\mathrm{i}g\theta^a(t)T^a\right). \tag{40}$$

The gauge transformed Hamiltonian degrees of freedom are then determined from

$$\begin{aligned}&A'^a T^a = UA^aT^aU^{-1} \qquad \pi'^a T^a = U\pi^aT^aU^{-1}\\ &\phi'^a T^a = U\phi^aT^aU^{-1} + \frac{\mathrm{i}}{g}U\frac{\mathrm{d}}{\mathrm{d}t}U^{-1}\,.\end{aligned} \tag{41}$$

Consequently, complete gauge fixing in this system is possible. Indeed, consider a certain configuration for (A^a, π^a, ϕ^a) and define the gauge transformation

$$U(t, t_0) = T\,\exp\left(-\mathrm{i}g\int_{t_0}^{t}\mathrm{d}t'\ \phi^a(t')T^a\right). \tag{42}$$

Then the transformed Lagrange multipliers vanish identically

$$\phi'^a(t) = 0 \tag{43}$$

while no additional gauge transformation exists which would leave this last identity invariant, given specific boundary conditions on A^a and/or π^a.

† In fact, the variables ϕ^a introduced here correspond to the opposite of the Lagrange multipliers λ^a introduced in section 2 in the general case, namely $\phi^a = -\lambda^a$.

Quantization and physical states

Let us now consider the quantized system. Given the expressions for H_0 and Q^a at the classical level, the corresponding operators are simply defined by

$$\hat{H}_0 = \tfrac{1}{2}\left[\left(\hat{\pi}^a\right)^2 + m^2\left(\hat{A}^a\right)^2\right] \tag{44}$$

and

$$\hat{Q}^a = gf^{abc}\hat{A}^b\hat{\pi}^c\,. \tag{45}$$

Due to the fundamental commutation relations

$$\left[\hat{A}^a(t), \hat{\pi}^b(t)\right] = \mathrm{i}\delta^{ab} \tag{46}$$

and the complete antisymmetry of the structure coefficients f^{abc}, the operators $\hat{H}_0$ and $\hat{Q}^a$ as defined above do not suffer quantum ordering ambiguities.

In view of the analogy with the ordinary harmonic oscillator, it is useful to introduce the Fock representation of the system, in terms of the operators

$$\alpha^a = \sqrt{\frac{m}{2}}\left[\hat{A}^a + \frac{\mathrm{i}}{m}\hat{\pi}^a\right] \qquad \alpha^{a\dagger} = \sqrt{\frac{m}{2}}\left[\hat{A}^a - \frac{\mathrm{i}}{m}\hat{\pi}^a\right] \tag{47}$$

or

$$\hat{A}^a = \frac{\alpha^a + \alpha^{a\dagger}}{\sqrt{2m}} \qquad \hat{\pi}^a = -\mathrm{i}\sqrt{\frac{m}{2}}\left[\alpha^a - \alpha^{a\dagger}\right] \tag{48}$$

such that

$$\left[\alpha^a, \alpha^{b\dagger}\right] = \delta^{ab}\,. \tag{49}$$

The Hamiltonian $\hat{H}_0$ then reads

$$\hat{H}_0 = \tfrac{1}{2}m\left[\alpha^a\alpha^{a\dagger} + \alpha^{a\dagger}\alpha^a\right] = m\left[\alpha^{a\dagger}\alpha^a + \tfrac{1}{2}D_G\right] \tag{50}$$

while the generators of the local gauge transformations become

$$\hat{Q}^a = -\mathrm{i}gf^{abc}\alpha^{b\dagger}\alpha^c\,. \tag{51}$$

Obviously, in particular one has

$$\left[\hat{Q}^a, \hat{Q}^b\right] = \mathrm{i}gf^{abc}\hat{Q}^c\,. \tag{52}$$

Given the normalized Fock vacuum $|0\rangle$

$$\alpha^a|0\rangle = 0 \qquad \langle 0|0\rangle = 1 \tag{53}$$

the orthonormalized basis of the Fock space, spanned by

$$|a_1, a_2, \ldots, a_n\rangle = N(a_1, a_2, \ldots, a_n)\,\alpha^{a_1\dagger}\alpha^{a_2\dagger}\cdots\alpha^{a_n\dagger}\,|0\rangle \tag{54}$$

where $N(a_1, a_2, \ldots, a_n)$ is a normalization factor, also diagonalizes the Hamiltonian $\hat{H}_0$ of the system, with

$$\hat{H}_0|a_1, a_2, \ldots, a_n\rangle = m(n + \tfrac{1}{2}D_G)|a_1, a_2, \ldots, a_n\rangle\,. \tag{55}$$

Note that this basis of orthonormalized states is in one-to-one correspondence with all fully symmetric irreducible representations of the unitary group $SU(D_G)$, whose Young tableaux reduce to single rows of all possible lengths ($n = 0, 1, \ldots$).

Consider now the subspace of physical states defined by the condition of local gauge invariance

$$\hat{Q}^a|\text{physical}\rangle = 0\,. \tag{56}$$

In view of the structure of the charges $\hat{Q}^a$, it is possible to show [5] that such states are necessarily all of the form

$$|n_1, \dots, n_\ell\rangle = N(n_1, \dots, n_\ell) \left[\mathrm{Tr}\left(\alpha^\dagger\right)^{r_1}\right]^{n_1} \cdots \left[\mathrm{Tr}\left(\alpha^\dagger\right)^{r_\ell}\right]^{n_\ell} |0\rangle \,. \tag{57}$$

Here, $N(n_1, \dots, n_\ell)$ are normalization factors whose evaluation has to be considered on a case-by-case basis for every choice of gauge group G, $n_1, \dots, n_\ell$ are arbitrary positive or vanishing integers, $r_1, \dots, r_\ell$ are the degrees of the independent invariant symmetric polynomials or Casimir operators in the group G of rank ℓ, and finally, the operators α and $\alpha^\dagger$ are defined by

$$\alpha = \alpha^a \, T^a \qquad \alpha^\dagger = \alpha^{a\,\dagger} \, T^a \tag{58}$$

the traces in (57) being taken in colour space only.

The orthonormalized states $|n_1, \dots, n_\ell\rangle$ are gauge singlets, as befits physical states, and span the entire space of physical states. In addition, they also diagonalize the Hamiltonian $\hat{H}_0$

$$\hat{H}_0 \, |n_1, \dots, n_\ell\rangle = m \left(n_1 r_1 + \cdots + n_\ell r_\ell + \tfrac{1}{2} D_G\right) \, |n_1, \dots, n_\ell\rangle \,. \tag{59}$$

In the simple case of $G = SU(2)$ of rank $\ell = 1$, it is straightforward to compute the normalization factor $N(n_1)$, in a manner which should be generalizable to an arbitrary group G. Let us introduce the operators

$$\begin{aligned} N &= \sum_{a=1}^{3} \alpha^{a\,\dagger} \alpha^a \qquad N^\dagger = N \\ B^\dagger &= \sum_{a=1}^{3} \alpha^{a\,\dagger} \alpha^{a\,\dagger} \qquad B = \sum_{a=1}^{3} \alpha^a \alpha^a \end{aligned} \tag{60}$$

whose algebra is simply

$$[N, B] = -2B \qquad \left[N, B^\dagger\right] = 2B^\dagger \qquad \left[B, B^\dagger\right] = 4N + 2D_G = 4N + 6 \,. \tag{61}$$

A simple calculation then leads to the following normalization of the basis $|n\rangle$ of the subspace of physical states

$$|n\rangle = \left[2^n \, n! \, \prod_{j=1}^{n} (2j + D_G - 2)\right]^{-1/2} \left(B^\dagger\right)^n \; |0\rangle \tag{62}$$

which thus satisfy the relations

$$\langle n|m\rangle = \delta_{n,m} \qquad n, m = 0, 1, \dots \,. \tag{63}$$

In particular, this result allows one to determine the configuration space wavefunction representation of physical states. These wavefunctions are defined by

$$\psi_n(A^a) \equiv \langle A^a|n\rangle \tag{64}$$

where $|A^a\rangle$ are the configuration space orthonormalized eigenstates of the operators $\hat{A}^a$. One then obtains

$$\begin{aligned} \psi_n(A^a) = {} & \left(\frac{m}{\pi}\right)^{D_G/4} \left[2^n \, n! \, (2n+1)!!\right]^{-1/2} \left(\frac{m}{2}\right)^n \left[\sum_{a=1}^{3} \left(A^a - \frac{1}{m} \frac{\partial}{\partial A^a}\right)^2\right]^n \\ & \times \exp\left(-\frac{1}{2} m \left(\sum_{a=1}^{3} A^a\right)^2\right) . \end{aligned} \tag{65}$$

Quite obviously, a similar analysis is possible in the general case of a specific but arbitrary gauge group G.

Physical time evolution of the quantum system

Let us now consider the physical time evolution of the system. According to the discussion of section 2, the corresponding operator is thus

$$S_{\rm phys}(t_2,t_1)=\exp\big(-{\rm i}\hat{H}_0\,(t_2-t_1)\big)\mathcal{E}=\mathcal{E}\ \exp\big(-{\rm i}\hat{H}_0(t_2-t_1)\big)\,\mathcal{E} \qquad (66)$$

where the projection operator $\mathcal{E}$ onto the subspace of physical states is defined by†

$$\mathcal{E}=\int {\rm d}U(\theta^a)\ \exp\big(-{\rm i}\theta^a\hat{Q}^a\big)\,. \qquad (67)$$

Here, ${\rm d}U(\theta^a)$ is the Haar measure over the gauge group G, the domain of integration being chosen according to the group G rather than its universal covering group $G_{\rm univ}$ when different. Once again, note that this measure is entirely specified by the requirement of the properties in (7) defining a projector, thereby avoiding at once both issues of gauge fixing and of the possibility of Gribov problems of the first or second type [11] related to a choice of gauge fixing‡.

Consider now the matrix element of the evolution operator between some initial and final states, $|\psi_i\rangle$ and $|\psi_f\rangle$ respectively, for a time interval $[t_i,t_f]$, namely

$$P(i\to f)=\langle\psi_f|\mathcal{E}\exp\big(-{\rm i}\hat{H}_0(t_f-t_i)\big)\mathcal{E}|\psi_i\rangle\,. \qquad (68)$$

As seen previously, the states $|a_1,a_2,\ldots,a_n\rangle$ in (54) span a complete orthonormalized basis of the space of states, including gauge variant ones, whereas the subset $|n_1,\ldots,n_\ell\rangle$ in (57) determines an orthonormalized basis of the space of physical states. Therefore, one may write

$$P(i\to f)=\sum_{n=0}^{\infty}\sum_{a_1,a_2,\ldots,a_n}\sum_{m=0}^{\infty}\sum_{b_1,b_2,\ldots,b_m}\langle\psi_f|a_1,a_2,\ldots,a_n\rangle$$
$$\times\langle a_1,a_2,\ldots,a_n|\mathcal{E}\exp\big(-{\rm i}\hat{H}_0(t_f-t_i)\big)\mathcal{E}|b_1,b_2,\ldots,b_m\rangle\langle b_1,b_2,\ldots,b_m|\psi_i\rangle\,. \qquad (69)$$

However, owing to the projection operators $\mathcal{E}$ to the left and to the right of the exponentiated Hamiltonian operator, only physical states do contribute to the sums over intermediate states. In addition, these physical states diagonalize the Hamiltonian $\hat{H}_0$, so that one finally obtains

$$P(i\to f)=\sum_{n_1,\ldots,n_\ell=0}^{\infty}\exp\Big(-{\rm i}m(n_1r_1+\cdots+n_\ell r_\ell+\tfrac{1}{2}D_G)(t_f-t_i)\Big)$$
$$\times\langle\psi_f|n_1,\ldots,n_\ell\rangle\langle n_1,\ldots,n_\ell|\psi_i\rangle\,. \qquad (70)$$

In conclusion, the physical evolution operator in (66) does indeed propagate as intermediate states physical states only, and in a manner which is consistent with the physical spectrum of the system. Moreover, any unphysical component of the external states, which thus has a vanishing overlap with the intermediate states $|n_1,\ldots,n_\ell\rangle$, is not propagated by the physical evolution operator. In fact, the matrix element $P(i\to f)$ vanishes identically whenever either one or both of the external states does not possess a gauge-invariant component. It is not that gauge variant components of states are not

† In the present case, the spectrum of $\hat{Q}^a$ being discrete, no δ-limiting procedure is required to properly define the reduced Hilbert space.

‡ Note that the converse result is established in [3], namely that the BRST invariant path integral for an admissible gauge fixing leads to the Haar measure over the gauge group for the Lagrange multipliers.

propagated in time in the system, but rather that the physical evolution operator in (66) does not propagate the gauge variant component of states.

Given the general result in (70), note also that it is possible in principle to compute any matrix element of the physical evolution operator (66), given the appropriate choice of initial and final states. For example, using the configuration space wavefunctions of physical states such as those given in (65) in the case of $SU(2)$, it is possible to obtain [5] the configuration space matrix elements of the physical evolution operator. Another possible choice is that of phase space coherent states.

Phase space coherent states

Finally, let us consider the phase space coherent states defined by

$$|\pi^a, A^b; \eta\rangle = \exp\big(\mathrm{i}\alpha(\pi^a, A^b)\big)\exp\big(-\mathrm{i}A^a\hat{\pi}^a\big)\exp\big(\mathrm{i}\pi^a\hat{A}^a\big)\,|\eta\rangle \tag{71}$$

where the choice of normalized fiducial state $|\eta\rangle$ is arbitrary, as well as the phase factor $\alpha(\pi^a, A^b)$.

Given the physical evolution operator in (66), its phase space coherent state matrix elements are given by

$$P(1\to 2) = \langle\pi_2, A_2; \eta|\mathcal{E}\exp\big(-\mathrm{i}\hat{H}_0(t_2-t_1)\big)\mathcal{E}|\pi_1, A_1; \eta\rangle\,. \tag{72}$$

Evaluation of this expression requires the calculation of the action of the projector $\mathcal{E}$ being applied to coherent states, and more specifically the result for

$$\exp\big(-\mathrm{i}\theta^a\hat{Q}^a\big)\,|\pi, A; \eta\rangle\,. \tag{73}$$

Given the parameters θ^a and the degrees of freedom π^a and A^a, let us define the quantities π^a_θ and A^a_θ by the relations

$$A^a_\theta T^a = UA^aT^aU^{-1} \qquad \pi^a_\theta T^a = U\pi^aT^aU^{-1} \tag{74}$$

where U is the finite gauge transformation in the group G:

$$U = \exp\Big(-\mathrm{i}\theta^aT^a\Big)\,. \tag{75}$$

Then, it is possible to show that one has

$$\exp\big(-\mathrm{i}\theta^a\hat{Q}^a\big)\,|\pi^a, A^b; \eta\rangle = |\pi^a_\theta, A^b_\theta; \eta_\theta\rangle \tag{76}$$

where the coherent state on the right-hand side is defined as in (71) with the fiducial state $|\eta_\theta\rangle$ now given by

$$|\eta_\theta\rangle = \exp\big(-\mathrm{i}\theta^a\hat{Q}^a\big)\,|\eta\rangle\,. \tag{77}$$

Consequently, the phase space coherent space matrix element in (72) takes the form

$$P(1\to 2) = \int \mathrm{d}U(\theta^a_2)\int \mathrm{d}U(\theta^a_1) \times\langle(\pi_2)^a_{\theta_2}, (A_2)^a_{\theta_2}; \eta_{\theta_2}|\exp\big(-\tfrac{1}{2}\mathrm{i}(t_2-t_1)[(\hat{\pi}^a)^2+m^2(\hat{A}^a)^2]\big)|(\pi_1)^a_{\theta_1}, (A_1)^a_{\theta_1}; \eta_{\theta_1}\rangle\,. \tag{78}$$

In view of the integration over the group parameters θ^a_1 and θ^a_2, it would be reasonable to believe that only gauge-invariant physical states contribute to this expression as intermediate states. As shown in (70), this is indeed the case, and the correct spectrum of physical states is in fact recovered from the time dependence of this expression.

We shall refrain here from computing (78) explicitly. This should be particularly simple [5] for the choice $|\eta\rangle = |0\rangle$, in which case $|\eta_\theta\rangle = |0\rangle$ as well. However, the expression in (70) seems to be better suited to the purpose of a calculation of (78), since one then only requires the overlap functions

$$\langle n_1, \ldots, n_\ell|\pi^a, A^b; \eta\rangle \tag{79}$$

of the phase space coherent states with the physical states $|n_1, \ldots, n_\ell\rangle$. These functions may be obtained using the Fock representation of the operators $\hat{A}^a$ and $\hat{\pi}^a$.

5. Conclusions

Within the phase space coherent state approach to quantization, Klauder's analysis [1] of the reproducing kernel or propagator for physical states in gauge-invariant systems is based on the physical projector [3–7] $\mathcal{E}$ onto the reduced Hilbert space of physical states. As shown in [1], the path integral measure for the Lagrange multipliers associated with the constraints is then uniquely determined from the projector property $\mathcal{E}^2 = \mathcal{E}$ of this operator, independently of any gauge-fixing conditions or reduction of second-class constraints. In addition, Klauder's approach does not require the introduction in the path integral representation of gauge-invariant observables of the δ-functionals and functional determinants which are typical of the conventional approaches to the quantization of constrained systems.

In the present paper, it is pointed out that since the physical propagator constructed on basis of that projection operator does not necessite gauge-fixing conditions, potential Gribov problems, which are typical of the conventional approaches to constrained systems, are avoided from the outset, while the properties of the physical projector $\mathcal{E}$ also ensure that the physical propagator does indeed lead to the correct physically consistent results for gauge-invariant observables, by effectively including once and only once the contribution from each of the inequivalent gauge orbits of the system, as would result from an admissible choice of gauge-fixing conditions in the conventional approaches. In other words, the role of the physical projector $\mathcal{E}$ is also to effectively determine the physically consistent integration measure over the modular space of the system, i.e. the quotient of configuration space or phase space, including Lagrange multiplier variables, by the gauge group. This important property of the physical propagator is confirmed explicitly in all these aspects by two simple examples, namely the free relativistic scalar particle and pure Yang–Mills theory in $0 + 1$ dimensions.

The analysis is performed within the abstract operator formulation of a quantized constrained system with first-class constraints only whose algebra is closed. Klauder's original reappraisal of the projected physical propagator is presented [1] within the context of the phase space coherent state path integral quantization of constrained systems. As is well know, the operator approach can be used to develop and justify the path integral one, thereby specifying unambiguously the integration measures over the phase space degrees of freedom and Lagrange multiplier variables in as far as the quantized system itself is uniquely and well defined at the operator level. In addition, the formulation of the physical projection operator $\mathcal{E}$ is such that manifest gauge invariance and covariance under other specific symmetries that the system may possess is maintained throughout. There is no need to develop a BRST description with its additional auxiliary and ghost degrees of freedom to achieve that aim, while the BRST approach may also be fraught with Gribov problems.

Clearly, based on the insight provided by Klauder's analysis, it would be extremely interesting to apply a similar approach to other gauge-invariant systems of physical interest, and see how the corresponding results compare with the understanding which has developed on the basis of the conventional approaches. Before considering more realistic theories in $3+1$ dimensions, obvious candidates would be the Yang–Mills and Chern–Simons theories, as well as quantum gravity theories in $1+1$ and $2+1$ dimensions, the quantum gravity theories in $1+1$ dimensions, including of course string theories.

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